Cumulants for Asymmetric Additive Convolution

V. Chub S. Surmylo

under guidance of C. Cuenca

Yulia's Dream Program

1. Background on probability and cumulants
2. Asymmetric additive convolution of polynomials
3. Main theorems

Random variable X:

$$P(X = x_1) = p_1, \\ \cdots \\ P(X = x_n) = p_n.$$

(Atomic) probability measure μ:

$$\mu = p_1\delta_{x_1} + \cdots + p_n\delta_{x_n}.$$

Moments of μ:

$$m_k(\mu) := \sum_{i=1}^{n} p_i x_i^k, \quad k = 1, 2, 3, \cdots$$

Uniform random variable X:

$$P(X = x_1) = p_1 = \frac{1}{n},$$

$$\cdots$$

$$P(X = x_n) = p_n = \frac{1}{n}.$$

Uniform probability measure:

$$\mu = \frac{1}{n}\sum_{i=1}^{n}\delta_{x_i} = \frac{1}{n}\delta_{x_1} + \cdots + \frac{1}{n}\delta_{x_n}.$$

Moments of a uniform probability measure:

$$m_k(\mu) := \frac{\sum_{i=1}^{n} x_i^k}{n}, \quad k = 1, 2, 3, \cdots$$

X, Y random variables are **independent** if

$$P(X=a,\ Y=b)=P(X=a)P(Y=b).$$

For a random variable X define the **cumulants** $\kappa_1(X), \kappa_2(X), \cdots$ as

$$\kappa_n(X)=n!\cdot[z^n]\ln\left(1+\sum_{n=1}^{\infty}\frac{z^n\,m_n(X)}{n!}\right), \tag{1}$$

where $m_n(X)$ $(n=\overline{1,\infty})$ are moments $m_n(\mu)$ of the probability measure μ of X.

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Theorem 1 (Linearity of cumulants)

For independent variables X, Y and all positive integers n:

$$\kappa_n(X + Y) = \kappa_n(X) + \kappa_n(Y)$$

Sketch of proof:

$$X, Y \text{ are independent} \implies \mathbb{E}[e^{z(X+Y)}] = \mathbb{E}[e^{zX}] \cdot \mathbb{E}[e^{zY}].$$

$$\text{By Taylor expansion: } \mathbb{E}[e^{zX}] = \mathbb{E}\left[1 + \sum_{n=1}^{\infty} \frac{z^n X^n}{n!}\right] = 1 + \sum_{n=1}^{\infty} \frac{z^n m_n(X)}{n!}.$$

$$\exp\left(\sum_{n=1}^{\infty} \frac{\kappa_n(X+Y) z^n}{n!}\right) = \exp\left(\sum_{n=1}^{\infty} \frac{\kappa_n(X) z^n}{n!}\right) \cdot \exp\left(\sum_{n=1}^{\infty} \frac{\kappa_n(Y) z^n}{n!}\right). \quad \square$$

Empirical root distribution

Consider a monic polynomial

$$P(x) = (x - r_1)(x - r_2) \cdots (x - r_d),$$

of degree d, where each root r_i is real. Its **empirical root distribution** is:

$$\mu^P := \frac{1}{d} \sum_{i=1}^{d} \delta_{r_i} = \frac{1}{d} \delta_{r_1} + \cdots + \frac{1}{d} \delta_{r_d}.$$

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An **even polynomial** is a polynomial on x^2 with **nonnegative** real roots, so any even polynomial $Q(x)$ can be expressed as

$$Q(x) = (x^2 - r_1^2)\cdots(x^2 - r_d^2) = (x - r_1)(x + r_1)\cdots(x - r_d)(x + r_d),$$

where $r_i \geq 0$. The empirical root distribution of Q is:

$$\mu^Q = \frac{1}{2d}\sum_{i=1}^{d}(\delta_{r_i} + \delta_{-r_i}),$$

and has moments:

$$m_k(\mu^Q) := \begin{cases} 0, & \text{for odd } k, \\ \frac{1}{d}\sum_{i=1}^{d} r_i^k, & \text{for even } k. \end{cases}$$

For monic polynomials $p(x)$ and $q(x)$ of degree d,

$$p(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^p, \qquad q(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^q,$$

where $a_0^p = a_0^q = 1$, [Marcus-Spielman-Srivastava '22] defined the **asymmetric additive convolution** $(p \boxplus_d^0 q)(x)$, as

$$(p \boxplus_d^0 q)(x) := \sum_{k=0}^{d} x^{d-k}(-1)^k \sum_{i+j=k} \left(\frac{(d-i)!(d-j)!}{d!(d-k)!} \right)^2 a_i^p a_j^q,$$

i.e.

$$(p \boxplus_d^0 q)(x) := x^d - \left(a_1^p + a_1^q\right)x^{d-1} + \left(a_2^p + \left(\frac{d-1}{d}\right)^2 a_1^p a_1^q + a_2^q\right)x^{d-2} - \cdots.$$

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Theorem 2

If $p(x)$ and $q(x)$ both have nonnegative real roots, then $(p \boxplus_d^0 q)(x)$ also has nonnegative real roots.

If $p(x) = x^d + a_1 x^{d-1} + \cdots + a_d$ has nonnegative real roots $r_1^2, \ldots, r_d^2$, for some $r_1, \ldots, r_d \geq 0$, then

$$p(x^2) = x^{2d} + a_1 x^{2d-2} + \cdots + a_d$$

has $(2d)$ real roots $\pm r_1, \ldots, \pm r_d$.

The **empirical root distribution**

$$\mu^p := \frac{1}{2d} \sum_{i=1}^{d} (\delta_{r_i} + \delta_{-r_i}).$$

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Probabilistic point of view: The polynomial operation $(p, q) \mapsto p \boxplus_d^0 q$ is “equivalent” to an operation $(\mu^p, \mu^q) \mapsto \mu^{p \boxplus_d^0 q}$.

This is similar to X, Y independent and $(X, Y) \mapsto X + Y$.

Our goal: define cumulants that “linearize” the asymmetric additive convolution.

Plan of the talk

Definition 3

For the monic polynomial $p(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^p$, with $a_0^p = 1$, define the **asymmetric cumulants** of $p(x)$ by:

$$\kappa_{2\ell}^p = 2\ell \cdot [y^{2\ell}] \ln \left(1 + \sum_{n=1}^{d} \left(\frac{(d-n)!}{2^n d!} \right)^2 a_n^p \, y^{2n} \right), \quad \text{for all } \ell = 1, 2, \ldots, d. \tag{2}$$

Theorem 4

For any monic polynomials $p(x), q(x)$ of degree d, we have

$$\kappa_{2k}^{p \boxplus_d^0 q} = \kappa_{2k}^{p} + \kappa_{2k}^{q}, \quad \text{for all } k = 1, 2, \ldots, d.$$

Thus, the asymmetric cumulants linearize asymmetric additive convolution.

Examples:

Recall: $(p \boxplus_d^0 q)(x) := x^d - \left(a_1^p + a_1^q\right)x^{d-1} + \left(a_2^p + \left(\frac{d-1}{d}\right)^2 a_1^p a_1^q + a_2^q\right)x^{d-2} - \cdots.$

For $k = 1$:

$$\kappa_2^p = \frac{a_1^p}{2d^2}, \qquad \kappa_2^q = \frac{a_1^q}{2d^2}, \qquad \kappa_2^{p \boxplus_d^0 q} = \frac{a_1^p + a_1^q}{2d^2} = \kappa_2^p + \kappa_2^q;$$

For $k = 2$:

$$\kappa_4^p = \frac{a_2^p}{4d^2(1-d)^2} - \frac{(a_1^p)^2}{8d^4}, \qquad \kappa_4^q = \frac{a_2^q}{4d^2(1-d)^2} - \frac{(a_1^q)^2}{8d^4},$$

$$\kappa_4^{p \boxplus_d^0 q} = \frac{a_2^p + \left(\frac{d-1}{d}\right)^2 a_1^p a_1^q + a_2^q}{4d^2(1-d)^2} - \frac{(a_1^p + a_1^q)^2}{8d^4} = \kappa_4^p + \kappa_4^q.$$

We want formulas between asymmetric cumulants and moments:

$$\kappa_2 = -\frac{m_2}{2d},$$
$$\kappa_4 = -\frac{m_4}{8d(d-1)^2} + \frac{2d-1}{8d^2(d-1)^2} m_2^2,$$

and

$$m_2 = -2d\kappa_2,$$
$$m_4 = -8d(d-1)^2\kappa_4 - 4d(2d-1)\kappa_2^2,$$
$$m_6 = -16d(d-1)^2(d-2)^2\kappa_6 + 16d(d-1)^2(7d-6)\kappa_4\kappa_2 - 8d(2d-1)^2\kappa_2^3.$$

Theorem 5

(i) For all $k \in \mathbb{Z}_{\geq 1}$*:*

$$\kappa_{2k} = \frac{1}{(2k-1)! \cdot 2^{2k}} \sum_{\sigma \in P^{even}(2k)} (-2d)^{\#(\sigma)} \prod_{B \in \sigma} (|B|-1)! \cdot m_\sigma \cdot \sum_{\pi :\ \pi \geq \sigma} \frac{(-1)^{\#(\pi)-1}(\#(\pi)-1)}{\prod_{D \in \pi} (-d)^2_{\frac{|D|}{2}}}.$$

(ii) For all $k \in \mathbb{Z}_{\geq 1}$*:*

$$m_{2k} = \frac{2^{2k-1}}{d \cdot (2k-1)!} \sum_{\sigma \in P^{even}(2k)} \prod_{B \in \sigma} (|B|-1)! \cdot \kappa_\sigma \cdot \sum_{\pi :\ \pi \geq \sigma} (-1)^{\#(\pi)} (\#(\pi)-1)! \prod_{D \in \pi} (-d)^2_{\frac{|D|}{2}}.$$

Thank you for attention!

Figure: https://www.britannica.com/science/mushroom